J. Fluid Mech. (1971), *vol.* 45, *part* 2, *pp.* 353–373
Printed in Great Britain

The asymptotic stability of a bounded rotating fluid heated from below: conductive basic state

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(Received 23 March 1970)

The asymptotic stability of a rapidly rotating, horizontally bounded fluid, heated from below, is treated using boundary-layer methods. It is shown that the rotational constraint is so strong as to preclude instabilities, if the interior regions of the fluid are considered to be inviscid. The correct formulation allows this constraint to be broken by introducing horizontal diffusive effects into the interior, while vertical diffusion is confined to Ekman layers on the horizontal surfaces; no vertical layers exist. Moreover, the mechanism of instability is (to the lowest order) associated with energy conversions entirely within the interior region. The present formulation elucidates the role of the Ekman layers in producing high-order corrections to the limiting critical Rayleigh number, and the asymptotic results are extended to include higher-order terms. The effect of rigid side walls on the critical Rayleigh number, and on the azimuthal wave-number, is considered. Except for very tall cylinders, the critical Rayleigh number is unaffected by the presence of side walls; the results for different azimuthal modes of convection are inconclusive, but indicate that no great error occurs if disturbances are assumed axisymmetric.

1. Introduction

The classical analysis of the stability of a fluid layer heated from below and subject to Coriolis force is due to Chandrasekhar (1953) and Chandrasekhar & Elbert (1955), and is summarized in Chandrasekhar (1961, chap. III). The fluid is assumed stagnant relative to a spinning frame and linearly stratified in the vertical. As has long been recognized, this is an acceptable approximation to the basic state if a suitable Froude number,

$$F = \omega^2 a_0/g, \tag{1.1}$$

is small. Here ω is the rotational frequency, a_0 a characteristic horizontal length, and g the acceleration of gravity. A linear stability analysis of the Boussinesq equations by means of normal modes yields Rayleigh number–wave-number relationships, with the Taylor number as a parameter. The problem is complicated by the fact that upon rotation oscillatory neutral states become possible; however, for Prandtl numbers greater than about 0·7, the onset of convection is stationary, and the critical Rayleigh number is independent of the Prandtl number. This

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criterion is provable for 'free–free' boundary conditions, and is approximately valid for the 'rigid–free' and 'rigid–rigid' cases (Finlayson 1968).

However, Veronis (1966, 1968) has cast doubt upon the validity of these linear results, especially in the oscillatory case. By the treatment of the full non-linear Boussinesq equations for a single convection cell, Veronis has shown that sub-linear finite amplitude instabilities may occur over a limited range of Prandtl and Taylor numbers, i.e. the predicted critical Rayleigh number is lower than that predicted by the linear theories. Use of a severely truncated model representation for the disturbances yielded steady sublinear instabilities for Prandtl numbers greater than $\sqrt{2}$ (Veronis 1966). Extensive calculations (Veronis 1968) indicate that the sublinear instability occurring at moderate Taylor numbers arises from a dominance of non-linear over Coriolis effects (a suitable Rossby number is large), while at higher Taylor numbers the rotational constraint becomes dominant. These effects are not observable for larger Prandtl numbers, since the scale of the motion decreases with increasing Prandtl number, such that an approximate thermal wind balance is achieved (Veronis 1968, p. 116). Accordingly, our linear treatment below will be assumed valid for $\sigma > O(2)$, say, for which onset is both stationary and predicted accurately by linear theory.

Linear stability calculations in the case of stationary onset yield asymptotic relations of the form,

$$\left.\begin{aligned} R &\rightarrow P_0 \tau^{\frac{2}{3}} \\ a &\rightarrow \alpha_0 \tau^{\frac{1}{6}} \end{aligned}\right\} \tau \rightarrow \infty. \tag{1.2}$$

Here $R = g\alpha\Delta T d^3/\nu\kappa$, and $\tau = 4\omega^2 d^4/\nu^2$ are the Rayleigh and Taylor numbers as usually defined, a is the dimensionless horizontal wave-number and P_0, α_0 are numerical constants. These constants were first believed to depend upon the boundary conditions at the horizontal surfaces (Chandrasekhar 1961, p. 104), but the asymptotic analysis of Niiler & Bisshopp (1965) has since proved that, to the lowest order,

$$\left.\begin{aligned} P_0 &= 3(\tfrac{1}{2}\pi^2)^{\frac{2}{3}} = 8{\cdot}6956, \\ \alpha_0 &= (\tfrac{1}{2}\pi^2)^{\frac{1}{6}} = 1{\cdot}305, \end{aligned}\right\} \tag{1.3}$$

results which are independent of the nature of the bounding surfaces. For rigid boundaries, the first correction to the asymptotic results (1.3) is of order $\tau^{-\frac{1}{12}}$; It was noted that this correction roughly accounts for the discrepancy between the asymptotic theory and the Rayleigh–Ritz calculations of Chandrasekhar.

In this paper we consider the stability of a radially bounded rotating cylinder of fluid. The basic state is stagnant with respect to a rotating co-ordinate system, unstably stratified, and the horizontal surfaces are rigid. The treatment will be asymptotic in the sense that the Taylor number will be assumed infinitely large. In a subsequent paper we consider the stability of a basic state which is not stagnant, i.e. we include the effect of motions produced by the centrifugal acceleration. As a necessary preparation for the treatment of more complicated basic states, we demonstrate here that boundary-layer methods may be applied successfully to the stability problem. Furthermore, the results for radially bounded fluids are shown to be similar in many respects to those for unbounded layers.

In § 2 the basic equations governing the linear stability of an initially stagnant fluid are presented in a form suitable for treatment of the asymptotic problem. The analysis opens in § 3 with a demonstration of the failure of the standard Ekman analysis to predict the stability characteristics of the fluid. Aided by consideration of the mechanical energy balance, it is shown that the rotational constraint precludes instability, and it is necessary to introduce the effects of horizontal diffusion into the interior regions in order to break this constraint. The equations are rescaled in § 4, and, by means of a slippery side-wall analysis, the asymptotic normal modes results are retrieved (§ 5) in a slightly more general form, and extended to include $O(\tau^{-\frac{1}{6}})$ correction terms. The mechanism of instability is shown to be energy conversions in the interior regions of the fluid. Lastly, in § 6, the effect of lateral walls on both the critical Rayleigh number and the preferred mode of convection is treated; it is shown that, although the axisymmetric mode is not clearly the least stable according to linear theory, no serious error in the Rayleigh number arises if the disturbances are assumed to be axisymmetric.

2. Basic equations

Consider a right circular cylinder of fluid having radius a_0 and depth d, rotating about its vertical axis with angular velocity ω. It is assumed that $F \ll 1$, and that centrifugal effects are negligible, so that the fluid is linearly stratified and stagnant with respect to a rotating reference frame. (For the effect of small but finite F on the basic state see Barcilon & Pedlosky (1967*c*).) Employing the Boussinesq approximation, and the equation of state

$$\rho = \rho_0(1-\alpha(T-T_0)), \tag{2.1}$$

the equations governing the linear stability of the fluid (assumed to have sufficiently high Prandtl number so that onset is steady) become

$$\nabla . \mathbf{q}' = 0, \tag{2.2a}$$

$$2\omega(\mathbf{k}\times\mathbf{q}') = -\rho_0^{-1}\nabla p' + g\alpha T'\mathbf{k} + \nu\nabla^2\mathbf{q}', \tag{2.2b}$$

$$-(\Delta T/d)(\mathbf{k}.\mathbf{q}') = \kappa\nabla^2 T'. \tag{2.2c}$$

Here g, α, κ, ν have their usual meaning, ΔT is the imposed temperature difference, ρ_0 the reference density, $\mathbf{q}'$ is the fluid velocity vector, T' the temperature perturbation from linear stratification, $\mathbf{k}$ the unit vector in the positive z direction, and ∇, ∇^2 are taken as three-dimensional operators in a rotating cylindrical polar co-ordinate system with origin at the lower bounding surface. p' is the reduced pressure,

$$p' = p + \rho_0 gz - \rho_0(\omega^2 r^2/2). \tag{2.3}$$

For rapidly rotating fluids, we take the scaling for the dimensionless (unprimed) quantities,

$$p' = \rho_0 d\alpha\Delta T g p, \quad \mathbf{q}' = (g\alpha\Delta T/2\omega)\mathbf{q},$$

$$T' = (\Delta T)T, \quad z = z'/d, \quad r = r'/d,$$

so that $0 \leqslant z \leqslant 1$, $0 \leqslant r \leqslant a_0/d \equiv r_0$, and the dimensionless formulation becomes

$$\nabla . \mathbf{q} = 0, \tag{2.4a}$$

$$\mathbf{k} \times \mathbf{q} = -\nabla p + \mathbf{k}T + E\nabla^2 \mathbf{q}, \tag{2.4b}$$

$$-\sigma S(\mathbf{k} . \mathbf{q}) = E\nabla^2 T, \tag{2.4c}$$

where $E = \nu/2\omega d^2$ (the Ekman number),

$\sigma = \nu/\kappa$ (the Prandtl number),

$S = \dfrac{g\alpha\Delta T}{4\omega^2 d}$ (the stratification parameter).

The Ekman number is defined so that

$$E^2 = \tau^{-1}, \tag{2.5}$$

thus facilitating comparison with previous results. The boundary conditions to be applied are

$$T = 0, \mathbf{q} = 0, \quad z = 0, 1, \tag{2.6}$$

$$\mathbf{q} = 0, \quad r = r, \tag{2.7}$$

and either

$$T = 0, \quad r = r_0, \quad \text{(conducting walls)}, \tag{2.8a}$$

or

$$\partial T/\partial r = 0, \quad r = r_0, \quad \text{(insulated walls)}. \tag{2.8b}$$

The quantity σS is sometimes referred to as the stratification (Barcilon & Pedlosky 1967*a*). Furthermore, except for factors of 2 and the minus sign in (2.4*c*) denoting unstable stratification, equations (2.4) are identical to those governing the motions of a rotating fluid at low Rossby number at conditions varying only slightly from linear stratification (Barcilon & Pedlosky 1967*a*, *b*; Greenspan 1968, p. 16). Although the set (2.4) describes a decidedly different physical situation than that considered by Barcilon & Pedlosky, there is a close mathematical analogy. We can therefore use the work of Barcilon & Pedlosky to indicate possible boundary-layer attacks. From this latter work it is known that solutions to the governing equations depend critically on the relation of σS to E as $E \to 0$.

σS is related to the Rayleigh number by

$$\sigma S = RE^2, \tag{2.9}$$

so that the asymptotic relation for unbounded layers, $R \to E^{-\frac{4}{3}}$, becomes $\sigma S = O(E^{\frac{2}{3}})$, $E \to 0$. If this relation is assumed to hold for radially bounded layers, and if we consider stratifications of $O(E^{\frac{2}{3}})$, we may draw the conclusion that a *standard* boundary-layer analysis (i.e. inviscid interior regions and interior length scales independent of E) would yield only the trivial solution. This is so because the lowest-order interior problem would not contain the effect of the stratification. This does not imply that $\sigma S = O(E^{\frac{2}{3}})$ is not the critical stratification for bounded layers, but only demonstrates the failure of the usual boundary-layer attack.

It is of interest to seek eigensolutions to the problem in the parameter space for which the lowest-order problem includes the non-homogeneity of the fluid, i.e. for the case,

$$\sigma S = O(E^{\frac{1}{2}}), \quad R \to E^{-\frac{3}{2}}. \tag{2.10}$$

We show below that although non-trivial solutions to (2.4) exist in this range, they yield no critical Rayleigh number in the usual sense. This seems to be a consequence of the fact that the isobars and isotherms of the present basic state are parallel.

3. Classical Ekman analysis, $\sigma S = O(E^{\frac{1}{2}})$

To begin, we consider regions away from solid surfaces to be inviscid, and (2.4) then yield for the interior variables,

$$\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial (ru)}{\partial r} = 0, \tag{3.1a}$$

$$v = \frac{\partial p}{\partial r}, \tag{3.1b}$$

$$u = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{3.1c}$$

$$0 = -\frac{\partial p}{\partial z}+T, \tag{3.1d}$$

$$-\sigma S w = E\nabla^2 T. \tag{3.1e}$$

In writing (3.1 e), we anticipate the fact that the horizontal Ekman layers entrain (expel) fluid with axial velocity $O(E^{\frac{1}{2}})$ for which convection will balance conduction. Substituting the geostrophic components (3.1 b, c) into the continuity equation yields the familiar result,

$$\frac{\partial w}{\partial z} = 0+O(E). \tag{3.2}$$

We also write from (3.1 c)

$$p = \int_0^z T\,dz + e(r,\theta), \tag{3.3}$$

where $e(r,\theta)$ is a function of integration. The geostrophic components u, v must be reduced to zero at $z = 0, 1$ by Ekman correction fields. The scaling in these layers is $u, v = O(1)$, $w = O(E^{\frac{1}{2}})$, $T = O(E)$. We do not give the details of the Ekman layer analysis, since the effect of these layers on the interior fields is adequately described by the Ekman compatibility relation which the interior axial velocity must satisfy. For the present case, this relation takes the form,

$$w|_{z=\frac{1}{2}\pm\frac{1}{2}} = \mp\frac{E^{\frac{1}{2}}}{\sqrt{2}}\zeta|_{z=\frac{1}{2}\pm\frac{1}{2}}. \tag{3.4}$$

(See, for example, Barcilon 1964, 1967.) Here $\zeta = \mathbf{k}.\nabla\times\mathbf{q}$ is the z component of the interior vorticity, and is related to the interior pressure p by

$$\zeta = \nabla_1^2 p, \tag{3.5}$$

where ∇_1^2 is the two-dimensional, horizontal Laplacian,

$$\nabla_1^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{3.6}$$

Applying (3.4) to the interior fields, we determine $\nabla_1^2 e$ and w,

$$\nabla_1^2 e = -\frac{1}{2}\int_0^1 \nabla_1^2 T\, dz, \tag{3.7a}$$

$$w = -\frac{E^{\frac{1}{2}}}{2\sqrt{2}}\int_0^1 \nabla_1^2 T\, dz. \tag{3.7b}$$

The energy equation then becomes the determining equation for the eigenvalue σS (equivalently, the critical Rayleigh number), viz.

$$\Lambda \int_0^1 \nabla_1^2 T\, dz = \nabla^2 T. \tag{3.8}$$

Here we have put $\Lambda = \sigma S E^{-\frac{1}{2}}/2\sqrt{2}$, and Λ becomes the $O(1)$ eigenvalue to be determined. The boundary conditions on the interior temperature are

$$T = 0, \quad z = 0, 1,$$

and, if the side walls are conducting,

$$T = 0, \quad r = r_0.$$

We do not write conditions if the walls are insulated, since this would require a detailed boundary-layer analysis near the side walls (see, for example, Homsy & Hudson 1969). A discussion of conducting side walls will suffice here. If we now put

$$T = Re(e^{im\theta} t(z) J_m(ar)), \tag{3.9}$$

where J_m is the mth-order Bessel function of the first kind, and a is a wave-number satisfying

$$J_m(ar_0) = 0,$$

the energy equation gives for $t(z)$,

$$-\Lambda a^2 \int_0^1 t(z)\, dz = \left(\frac{d^2}{dz^2} - a^2\right) t(z), \tag{3.10a}$$

$$t = 0, \quad z = 0, 1. \tag{3.10b}$$

Letting

$$K = \int_0^1 t(z)\, dz \quad \text{(a constant)},$$

eigensolutions of (3.10) are simply

$$t(z) = \frac{\Lambda K(\cosh(a) - 1)}{\sinh(a)} \sinh(az) - \Lambda K \cosh(az) + \Lambda K. \tag{3.11}$$

Inserting (3.11) into the definition of K yields the eigenvalue (Rayleigh number–wave-number) relation,

$$\Lambda(a) = \frac{a}{a - 2\tanh(a/2)}, \tag{3.12}$$

which is seen to have no extremum, $0 < a < \infty$; thus, we conclude that no critical Rayleigh number exists in the parameter region,

$$R = O(E^{-\frac{2}{3}}), \quad E \to 0.$$

The reason for this can be seen as follows; the potential energy of the fluid due to its unstable stratification must be released in order for onset of convection to occur. This release occurs when a Rayleigh number is reached such that the rate of viscous dissipation of kinetic energy balances the rate of release of potential energy (Chandrasekhar 1961, p. 130). For the present case, we formulate this balance by taking the vector product of the velocity with the equations of motion and integrating over the volume, viz.

$$\int_V \mathbf{q}\cdot(\mathbf{k}\times\mathbf{q})\,dV = \int_V (-\mathbf{q}\cdot\nabla p + \mathbf{q}\cdot\mathbf{k}T + E\mathbf{q}\cdot\nabla^2\mathbf{q})\,dV. \tag{3.13}$$

Noting that $\qquad \mathbf{q}\cdot\mathbf{k}\times\mathbf{q} = 0 \quad \text{and} \quad \int_V \mathbf{q}\cdot\nabla p\,dV = 0$

for solid bounding surfaces, we have

$$0 = \int_V wT\,dV + E\int_V \Phi\,dV, \tag{3.14}$$

where Φ is the dissipation function, which for the present case may be written

$$\Phi = 2[(u_r)^2 + (r^{-1}(v_\theta + u))^2 + (w_z)^2] + [r(r^{-1}v)_r + r^{-1}u_\theta]^2 + [r^{-1}w_\theta + v_z]^2 + [u_z + w_r]^2. \tag{3.15}$$

The first term in (3.14) represents the rate of release of potential energy and the second is of course the viscous dissipation of kinetic energy. For instability to occur there must be a balance between them.

For the solution just obtained, there can be no balance in the interior since

$$\int_V wT\,dV = O(E^{\frac{1}{2}}),$$

while $$E\int_V \Phi\,dV = O(E)$$

there. However, due to the Ekman length scale, it is true that the dissipation,

$$E\int_V \Phi\,dV = O(E^{\frac{1}{2}}),$$

in the Ekman layers (see, for example, Barcilon 1964, p. 298). This possible balance between the rate of release of potential energy in the interior and viscous dissipation in the Ekman layers does not give rise here to an instability as it did in the baroclinic stability problem in an annulus treated by Barcilon (1964). This is apparently the case because the mechanism of instability differs in the two problems. In the baroclinic stability problem, the instability occurs because a fluid particle travelling vertically moves between isobars and isotherms which enables it to release its potential energy. For the present problem, the isotherms and isobars are essentially parallel (consider (2.3) with $F \ll 1$); thus, there is no 'baroclinicity' in the basic state and this mode of instability is absent. Parenthetically, the fact that this possible balance did not produce an instability is an indication that the Ekman layers will have a higher-order effect on the stability characteristics of the fluid; this is confirmed in the next section.

It appears, then, that the rotational constraint together with the assumption of $O(1)$ horizontal, as well as vertical variations in the interior, combine to preclude any instabilities in this parameter region. In the following section, we construct a solution for which the necessary balance in (3.14) is effected in the bulk of the fluid. As a final remark, we note that the governing equations (3.1) are hydrostatic in the vertical; hence, any temperature perturbation is balanced by a vertical pressure gradient, and any instabilities are effectively suppressed. Not surprisingly, the correct asymptotic treatment of the next section relaxes this hydrostatic restraint by introducing the effects of diffusion into the inteiior.

4. Ekman analysis with $O(E^{-\frac{1}{3}})$ horizontal variation

We will now treat the stability problem under the assumption that the horizontal variation of disturbances is dimensionlessly

$$\partial/\partial r \sim E^{-\frac{1}{3}}, \quad E \to 0,$$

and, together with a modified Ekman analysis, we will obtain a formulation of the stability problem which is uniformly valid as $E \to 0$. We are led to pick this horizontal variation, since it is suggested by the normal modes results for an infinitive layer, i.e.

$$\partial/\partial r \sim a \sim E^{-\frac{1}{3}}.$$

Furthermore, numerical results for finite Taylor numbers in a bounded cylinder (Homsy 1969), indicate that: (i) the horizontal scale of the motion decreases with decreasing E, and (ii) no regions of sharp horizontal gradients (vertical boundary layers) form as E decreases. Lastly, the rotational constraint requires $\partial/\partial z = O(1)$, so that the only apparent way to satisfy the mechanical energy balance is to relax the condition $\partial/\partial r = O(1)$.

We thus define a new dimensionless co-ordinate

$$x = r'/(dE^{\frac{1}{3}}),$$

which has limits $0 \leqslant x \leqslant r_0 E^{-\frac{1}{3}}$. Furthermore, following Niiler & Bisshopp, we put

$$R = PE^{-\frac{4}{3}},$$

where P is now the $O(1)$ constant eigenvalue whose determination yields the asymptotic dependence of R on E. With the new scaling, (2.4) become

$$(\mathbf{k}.\mathbf{q})_z + E^{-\frac{1}{3}}\nabla_1.\mathbf{q} = 0, \tag{4.1a}$$

$$\mathbf{k}\times\mathbf{q} = -E^{-\frac{1}{3}}\nabla_1 p - \mathbf{k}p_z + E^{\frac{1}{3}}\nabla_1^2\mathbf{q} + E\mathbf{q}_{zz} + \mathbf{k}T, \tag{4.1b}$$

$$-PE^{\frac{2}{3}}(\mathbf{k}.\mathbf{q}) = E^{\frac{1}{3}}\nabla_1^2 T + ET_{zz}. \tag{4.1c}$$

Here

$$\nabla_1 = \left(\frac{\partial}{\partial x}, \frac{1}{x}\frac{\partial}{\partial \theta}, 0\right), \tag{4.2a}$$

and

$$\nabla_1^2 = x^{-1}\frac{\partial}{\partial x}x\frac{\partial}{\partial x} + x^{-2}\frac{\partial^2}{\partial \theta^2}. \tag{4.2b}$$

We now expand the interior dependent variables and the eigenvalue in an asymptotic series in $E^{\frac{1}{6}}$, viz.

$$\left.\begin{aligned} \mathbf{q} &= \mathbf{q}_0 + E^{\frac{1}{6}}\mathbf{q}_1 + E^{\frac{1}{3}}\mathbf{q}_2 + \dots, \\ T &= E^{\frac{1}{3}}T_2 + E^{\frac{1}{2}}T_3 + \dots, \\ p &= E^{\frac{1}{3}}p_2 + E^{\frac{1}{2}}p_3 + \dots, \\ P &= P_0 + E^{\frac{1}{6}}P_1 + \dots. \end{aligned}\right\} \tag{4.3}$$

We anticipate powers of $E^{\frac{1}{6}}$, that being the common factor between the interior horizontal scale and the vertical Ekman scale. The expansion (4.3) then yields the sets,

$$x^{-1}v_{0,\theta} + x^{-1}(xu_0)_x = 0, \tag{4.4a}$$

$$v_0 = p_{2,x}, \quad u_0 = -x^{-1}p_{2,\theta}, \tag{4.4b, c}$$

$$0 = -p_{2,z} + \nabla_1^2 w_0 + T_2, \tag{4.4d}$$

$$-P_0 w_0 = \nabla_1^2 T_2, \tag{4.4e}$$

$$x^{-1}v_{1,\theta} + x^{-1}(xu_1)_x = 0, \tag{4.5a}$$

$$v_1 = p_{3,x}, \; u_1 = -x^{-1}p_{3,\theta}, \tag{4.5b, c}$$

$$0 = -p_{3,z} + \nabla_1^2 w_1 + T_3, \tag{4.5d}$$

$$-(P_0 w_1 + P_1 w_0) = \nabla_1^2 T_3. \tag{4.5e}$$

Since (v_0, u_0), (v_1, u_1) are in geostrophic balance, the continuity equation is degenerate, and we must continue the expansion to $O(E^{\frac{1}{3}})$ to obtain an equation relating w_0 and p_2. These $O(E^{\frac{1}{3}})$ equations are

$$w_{0,z} = -(x^{-1}v_{2,\theta} + x^{-1}(xu_2)_x), \tag{4.6a}$$

$$v_2 = p_{4,x} - \nabla_1^2\mathbf{q}_0)_x, \tag{4.6b}$$

$$u_2 = -x^{-1}p_{4,\theta} + \nabla_1^2\mathbf{q}_0)_\theta. \tag{4 .6}$$

Here $)_x$ and $)_\theta$ denote the x and θ components of the vector $\nabla_1^2\mathbf{q}_0$, respectively. Eliminating u_0 and v_0 between $(4.6a, b, c)$ and $(4.4b, c)$ leads to the required relation,

$$\partial w_0/\partial z = -\nabla_1^4 p_2. \tag{4.7}$$

A similar relation holds for w_1, i.e.

$$\partial w_1/\partial z = -\nabla_1^4 p_3. \tag{4.8}$$

Closed sets for the first- and second-order eigenvalue problems can now be written, viz.

$$\frac{\partial w_0}{\partial z} = -\nabla_1^4 p_2, \tag{4.9a}$$

$$0 = -\frac{\partial p_2}{\partial z} + \nabla_1^2 w_0 + T_2, \tag{4.9b}$$

$$-P_0 w_0 = \nabla_1^2 T_2; \tag{4.9c}$$

$$\frac{\partial w_1}{\partial z} = -\nabla_1^4 p_3, \tag{4.10a}$$

$$0 = -\frac{\partial p_3}{\partial z} + \nabla_1^2 w_1 + T_3, \tag{4.10b}$$

$$-(P_0 w_1 + P_1 w_0) = \nabla_1^2 T_3. \tag{4.10c}$$

It is seen that the assumed horizontal scale has resulted in an interior problem of sufficiently high order to satisfy the conditions at the side walls, and thus as indicated above, we expect no vertical boundary layers. The systems (4.9), (4.10) are only of second order in z however, and we cannot satisfy all of

$$w = v = u = T = 0, \quad z = 0, 1.$$

In order to derive proper conditions for the interior fields, it is then necessary to consider the Ekman layers near $z = 0, 1$. Denoting the Ekman correction fields with a tilde, we write

$$\mathbf{q} = \mathbf{q}(x, \theta, z) + \tilde{\mathbf{q}}(x, \theta, \zeta), \tag{4.11a}$$

$$T = T(x, \theta, z) + \tilde{T}(x, \theta, \zeta), \tag{4.11b}$$

$$p = p(x, \theta, z) + \tilde{p}(x, \theta, \zeta), \tag{4.11c}$$

where ζ is the Ekman co-ordinate

$$\zeta = \zeta_0 = z/(2E)^{\frac{1}{2}}$$

near $z = 0$, and

$$\zeta = \zeta_1 = (z-1)/(2E)^{\frac{1}{2}}$$

near $z = 1$. Thus, $0 \leqslant \zeta_0 \leqslant \infty$, $-\infty \leqslant \zeta_1 \leqslant 0$. The equations governing the tilde fields (near either $z = 0$ or $z = 1$) become

$$\frac{1}{\sqrt{2}}(\mathbf{k}\cdot\tilde{\mathbf{q}})_\zeta + E^{\frac{1}{6}}(\boldsymbol{\nabla}_1\cdot\tilde{\mathbf{q}}) = 0, \tag{4.12a}$$

$$(\mathbf{k}\times\tilde{\mathbf{q}}) = -(2E)^{-\frac{1}{2}}(\mathbf{k}\tilde{p}_\zeta) - E^{-\frac{1}{3}}(\boldsymbol{\nabla}_1\tilde{p}) + E^{\frac{1}{3}}\nabla_1^2\tilde{\mathbf{q}} + \tfrac{1}{2}\tilde{\mathbf{q}}_{\zeta\zeta} + \mathbf{k}\tilde{T}, \tag{4.12b}$$

$$-PE^{\frac{2}{3}}(\mathbf{k}\cdot\tilde{\mathbf{q}}) = E^{\frac{1}{3}}\nabla_1^2\tilde{T} + \tfrac{1}{2}\tilde{T}_{\zeta\zeta}. \tag{4.12c}$$

Again we expand

$$\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_0 + E^{\frac{1}{6}}\tilde{\mathbf{q}}_1 + \dots, \tag{4.13a}$$

$$\tilde{T} = \tilde{T}_0 + E^{\frac{1}{6}}\tilde{T}_1 + \dots, \tag{4.13b}$$

$$\tilde{p} = E^{\frac{1}{2}}\tilde{p}_3 + \dots. \tag{4.13c}$$

For the first order, we have

$$\tilde{w}_{0,\zeta} = 0, \tag{4.14a}$$

$$-2\tilde{v}_0 = \tilde{u}_{0,\zeta\zeta} \tag{4.14b}$$

$$2\tilde{u}_0 = \tilde{v}_{0,\zeta\zeta} \tag{4.14c}$$

$$0 = -\frac{1}{\sqrt{2}}\tilde{p}_{3,\zeta} + \tfrac{1}{2}\tilde{w}_{0,\zeta\zeta} + \tilde{T}_0, \tag{4.14d}$$

$$\tilde{T}_{0,\zeta\zeta} = 0. \tag{4.14e}$$

From (4.14a, d, e) we obtain $\tilde{w}_0 = \tilde{p}_0 = \tilde{T}_0 = 0$ for both top and bottom layers. Equations (4.14b, c) yield the usual Ekman spiral solution, which when required to reduce the interior components to zero at the surfaces yields

$$\tilde{u}_0(x, \theta, \zeta_0) = -e^{-\zeta_0}(u_0(x, \theta, 0)\cos\zeta_0 + v_0(x, \theta, 0)\sin\zeta_0), \tag{4.15a}$$

$$\tilde{v}_0(x, \theta, \zeta_0) = -e^{-\zeta_0}(v_0(x, \theta, 0)\cos\zeta_0 - u_0(x, \theta, 0)\sin\zeta_0), \tag{4.15b}$$

with similar expressions near the top.

At this point we note that, from (4.12c) $\tilde{T}_1 = \tilde{T}_2 = 0$, and the above analysis suffices to set boundary conditions for the first-order interior problem, (4.9). We obtain these from the original conditions as

$$w_0 = T_2 = 0, \quad z = 0, 1. \tag{4.16}$$

These conditions are not independent, however, in light of (4.9*c*), and we take as our conditions

$$w_0 = 0, \quad z = 0, 1. \tag{4.17}$$

These are the same conditions arrived at heuristically by Chandrasekhar (1961, p. 104); and they were shown to be the correct ones *a posteriori* by the asymptotic analysis of Niiler & Bisshopp.

Before discussing solutions to the first-order problem, we complete the specification of the second-order problem by considering the second-order Ekman equations,

$$\tilde{w}_{1,\zeta} = -\sqrt{2}x^{-1}(\tilde{v}_{0,\theta} + (x\tilde{u}_0)_x), \tag{4.18a}$$

$$-2\tilde{v}_1 = \tilde{u}_{1,\zeta\zeta}, \quad 2\tilde{u}_1 = \tilde{v}_{1,\zeta\zeta}, \tag{4.18b,c}$$

$$0 = -\frac{1}{\sqrt{2}}\tilde{p}_{4,\zeta} + \tfrac{1}{2}\tilde{w}_{1,\zeta\zeta} + \tilde{T}_1, \tag{4.18d}$$

$$\tilde{T}_{1,\zeta\zeta} = 0. \tag{4.18e}$$

Here we have used the fact that $\tilde{p}_3 = 0$. As noted above, $\tilde{T}_1 = 0$, but because of (4.18*a*), the $O(E^{\frac{1}{6}})$ vertical velocity in these layers is not zero, but is equal to the usual Ekman suction velocity modified in magnitude due to the $O(E^{\frac{1}{3}})$ horizontal variation. Thus, an integration of (4.18*a*) for the top and bottom layers then produces the modified Ekman compatibility condition which the $O(E^{\frac{1}{6}})$ axial velocity must satisfy, viz.

$$w_1\big|_{z=\frac{1}{2}\pm\frac{1}{2}} = \mp\frac{\nabla_1^2 p_2}{\sqrt{2}}\bigg|_{z=\frac{1}{2}\pm\frac{1}{2}}. \tag{4.19}$$

Thus, w_1, unlike w_0, does not vanish at the boundaries, but in fact must balance the small $O(E^{\frac{1}{6}})$ flux required by the convergence of the first-order Ekman layers. Lastly, we note that (4.18*b, c*) are of sufficient order to reduce the second-order interior fields to zero at the boundaries.

It is of interest to demonstrate that the problem posed with the above scaling is capable of satisfying the mechanical energy balance in the interior alone. For the present formulation, $w = O(1)$, $T = O(E^{\frac{1}{3}})$ in the interior, so that

$$\int_V wT\,dV = O(E^{\frac{1}{3}}) \tag{4.20a}$$

there. For the dissipation, we now recognize that interior motions have a new horizontal scale, while the interior velocity itself is $O(1)$. Thus, from only *certain terms* in Φ,

$$E\int_V \Phi\,dV = E\int_V E^{-\frac{2}{3}}\left(\frac{\partial q}{\partial x}\right)^2 dV = O(E^{\frac{1}{3}}) \tag{4.20b}$$

due to the new scale, x. A balance is now clearly possible, and the instability is shown to be due to energy conversions in the *interior* region of the fluid. As mentioned above, it was necessary to introduce diffusional effects into the interior, so that the vertical momentum balance is no longer hydrostatic (see (4.9*b*)). Indeed, this is the only way in which the rotational constraint (i.e. the fact that axial variations and axial velocities are inhibited by rotation) can be relaxed, so that the energy conversion required for instability may take place. We make heavy use of these facts in a subsequent paper, where an understanding of the above formulation will be necessary.

5. Stability results for slip walls

It is of interest to consider the stability problem for slippery side walls for three reasons. First, we wish to ensure that the boundary-layer analysis will yield the correct asymptotic results obtained for normal modes. Once having accomplished this, we then use our boundary-layer formulation to extend the results to $O(E^{\frac{1}{3}})$. Secondly, we demonstrate that asymmetric three-dimensional disturbances are valid planform functions, giving results equivalent to normal modes. This does not appear to have been shown previously, although the axisymmetric demonstration is due to Müller (1965). Thirdly, the above formulation affords an explicit delineation of the role of the Ekman layers in producing $O(E^{\frac{1}{6}})$ corrections to the asymptotic results.

We begin by combining (4.9) into a single equation for w_0, viz.

$$P_0 \nabla_1^2 w_0 = \frac{\partial^2 w_0}{\partial z^2} + \nabla_1^6 w_0, \tag{5.1a}$$

$$w_0 = 0, \quad z = 0,1. \tag{5.1b}$$

For slippery side walls, we drop the boundary conditions at $x = r_0 E^{-\frac{1}{3}}$, and separable solutions to (5.1*a*, *b*) are then

$$w_0 = \mathrm{Re}\,(2^{\frac{1}{2}} \sin(n\pi z)\, e^{im\theta} J_m(\alpha x)), \tag{5.2}$$

if P_0 satisfies the eigenvalue relation

$$P_0 = \frac{(n\pi)^2 + \alpha^6}{\alpha^2}. \tag{5.3}$$

Here α is related to the wave-number a as

$$a = \alpha E^{-\frac{1}{3}}. \tag{5.4}$$

The relation (5.3) is of course the same as that obtained from normal modes (Niiler & Bisshopp 1965, p. 756), and attains its minimum at $n = 1$;

$$\alpha_c^6 = \tfrac{1}{2}\pi^2,$$

$$\alpha_c = 1{\cdot}305, \tag{5.5}$$

for which

$$P_0 = 3\alpha_c^4 = 8{\cdot}6956. \tag{5.6}$$

Furthermore, the form (5.2) of the disturbance may be interpreted as describing n circulation cells having azimuthal wave-number m. By relating w_0 to u_0 via the continuity equation, the number of cells is related to the aspect ratio of a cylinder having slippery insulated sidewalls as

$$r_0 E^{-\frac{1}{3}} = \alpha_n^1 / 1{\cdot}305 \quad (m = 0), \tag{5.7a}$$

$$r_0 E^{-\frac{1}{3}} = \alpha_n^m / 1{\cdot}305 \quad (m > 0), \tag{5.7b}$$

where α_n^m is the nth zero of J_m, i.e.

$$J_m(\alpha_n^m) = 0 \quad (n = 1, 2, \ldots). \tag{5.8}$$

We now consider the second-order eigenvalue problem posed by (4.10) for the case of slip walls. Eliminating p_3, T_3 we have

$$P_0 \nabla_1^2 w_1 + P_1 \nabla_1^2 w_0 = \frac{\partial^2 w_1}{\partial z^2} + \nabla_1^6 w_1, \tag{5.9a}$$

$$w_1|_{z=\frac{1}{2}\pm\frac{1}{2}} = \mp \nabla_1^2 p_2/\sqrt{2}|_{z=\frac{1}{2}\pm\frac{1}{2}}. \tag{5.9b}$$

Using our results for w_0, p_2, (5.9*b*) may be written

$$w_1|_{z=\frac{1}{2}\pm\frac{1}{2}} = \mp \frac{1}{\sqrt{(2)}\alpha^2} \frac{\partial w_0}{\partial z}\bigg|_{z=\frac{1}{2}\pm\frac{1}{2}}. \tag{5.9c}$$

It is convenient to introduce the notation

$$w_0 = \mathrm{Re}\,(W_0(z) e^{im\theta} J_m(\alpha x)),$$
$$w_1 = \mathrm{Re}\,(W_1(z) e^{im\theta} J_m(\alpha x)),$$

where of course $W_0(z) = 2^{\frac{1}{2}} \sin(\pi z)$. Equation (5.9*a*) then yields for W_1

$$W_1'' - \alpha^6 W_1 + \alpha^2 (P_0 W_1 + P_1 W_0) = 0. \tag{5.10}$$

To obtain the eigenvalue P_1, we multiply (5.10) by W_0, integrate over z, and integrate by parts when necessary; in addition, we use the facts that

$$W_0'' = -(P_0 \alpha^2 + \alpha^6) W_0$$

and that W_1 does not vanish on the boundaries while W_0 does. In this manner, we obtain

$$P_1 \alpha^2 = -W_1 W_0' |_{z=0}^{1}. \tag{5.11}$$

This relation makes clear the fact that $O(E^{\frac{1}{6}})$ corrections to (5.6) are a direct consequence of the dissipation in the first-order Ekman layers, as evidenced by our modified Ekman conditions (5.9*b*). If this effect were neglected (or of higher order, e.g. near a free surface), the immediate consequence would be $P_1 = 0$.

Further manipulation of (5.11) yields

$$\begin{aligned} P_1 \alpha^2 &= -\frac{1}{\sqrt{(2)}\alpha^2} [(W_0'(1))^2 + (W_0'(0))^2] \\ &= -4\pi^2/\sqrt{(2)}\alpha^2. \end{aligned} \tag{5.12}$$

Thus, through first order,

$$P\alpha^2 = \pi^2 + \alpha^6 - \frac{4\pi^2}{\sqrt{(2)}\alpha^2} E^{\frac{1}{6}}, \tag{5.13}$$

which is in exact agreement with the normal modes results (Niiler & Bisshopp 1965, p. 757). We take that as a demonstration of the validity of the boundary-layer formulation.

We have also extended the analysis in a straightforward manner through $O(E^{\frac{1}{3}})$. The details of the computation are given in the appendix. The results are

$$P\alpha^2 = \pi^2 + \alpha^6 - \frac{4\pi^2}{2\alpha^2} E^{\frac{1}{6}} + \frac{6\pi^2}{\alpha^4} E^{\frac{1}{3}}. \tag{5.14}$$

Requiring P to be a minimum yields the computational formulae,

$$P = 8{\cdot}6956\,(1. - 1{\cdot}108E^{\frac{1}{6}} + 0{\cdot}1533E^{\frac{1}{3}}), \qquad (5.15)$$

$$\alpha = 1{\cdot}305\,(1. - 0{\cdot}5538E^{\frac{1}{6}} - 0{\cdot}3450E^{\frac{1}{3}}). \qquad (5.15b)$$

The progression of the numerical constant in (5.15a) indicates that the 'rough' agreement between the results of Niiler & Bisshopp and Chandrasekhar at $E = 10^{-5}$ is not merely qualitative.

In table 1 we show the quantity P/P_0 as calculated using (5.13), (5.15a), and the Rayleigh–Ritz method. At $E = 10^{-5}$, the agreement of the asymptotic results with those of Chandrasekhar is good ($< 2\%$ difference), but the extension of (5.15a) to higher Ekman numbers would seem to require the calculation of additional terms. The agreement between the value of α at $E = 10^{-5}$ using (5.15b) and that of Chandrasekhar is excellent. Thus, (5.15) provide accurate asymptotic formulae and combine with previous results to yield stability characteristics for the entire range of Ekman (Taylor) numbers.

	P/P_0		
E	2 terms	3 terms	Rayleigh–Ritz
0	1·0	1·0	—
10^{-6}	0·889	0·891	—
10^{-5}	0·837	0·841	0·857
10^{-4}	0·761	0·768	0·817

TABLE 1. The quantity P/P_0 calculated by different methods

6. The effect of side walls

We now supplement the normal modes results by treating the asymptotic equations including the effect of rigid side walls. This is of little practical importance since it will be seen from the results below that for small Ekman number, most cylinders of reasonable shape (excluding very tall cylinders) can be regarded as unbounded layers as far as stability characteristics are concerned. (This statement is true, however, only if centrifugal effects are ignored.) What follows is of theoretical interest, however, since it affords an opportunity to investigate the effect of side walls on the preferred mode of instability, allowing fully three-dimensional disturbances. Extensive experimental evidence for both stationary and rotating systems (e.g. Koschmieder 1966, 1967a, b, 1968; Liang, Vidal & Acrivos 1969) has indicated that the initial motion is greatly influenced by container shape, although super-critical motions are influenced by non-linear and second-order effects, such as finite-amplitude motions, temperature dependence of viscosity, free surface deflections, etc. A theoretical *linear* treatment of convection in a box (due to Davis 1967) succeeded in specifying preferred cell *orientations*, but the results were limited to two-dimensional rolls. Below we treat three-dimensional disturbances.

It is convenient to rescale the radial co-ordinate so that it ranges from 0 to 1. Thus, if we introduce

$$r = r'/a_0,$$

(4.9) then yield (dropping the subscripts on dependent variables)

$$\frac{\partial w}{\partial z} = -x_0^{-4}\nabla_1^4 p, \tag{6.1a}$$

$$\frac{\partial p}{\partial z} = x_0^{-2}\nabla_1^2 w + T, \tag{6.1b}$$

$$-Pw = x_0^{-2}\nabla_1^2 T. \tag{6.1c}$$

Here $0 \leqslant r \leqslant 1, x_0 = r_0 E^{-\frac{1}{3}}$. The boundary conditions for these *interior* fields are

$$w = 0, \quad z = 0,1, \tag{6.2a}$$

$$w = \frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial r} = 0, \quad (r = 1), \tag{6.2b}$$

together with $$T = 0, \quad (r = 1) \text{ (conducting walls)}, \tag{6.2c}$$

or $$\frac{\partial T}{\partial r} = 0, \quad (r = 1) \text{ (insulated walls)}. \tag{6.2d}$$

If the motion is taken as axisymmetric, the radial component of velocity u falls in magnitude from $O(1)$ to $O(E^{\frac{1}{3}})$, and (6.2b) must be replaced by

$$w = \frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial^2 p}{\partial r^2}\right) = 0, \quad (r = 1). \tag{6.2e}$$

This last condition follows from a consideration of (4.4b, 4.6c) for axisymmetry. We also note that in deriving (6.1) horizontal diffusive effects were present in the interior of the fluid *with the exclusion of all* vertical diffusion, the latter being confined to Ekman layers. This is valid only if $x_0 \gg 1$, with an incurred error of $O(x_0^{-2})$. Hence our treatment of (6.1) below for finite x_0 is not strictly valid, but we restrict our attention to $x_0 > 10{\cdot}0$, for which we expect the results to be reasonably accurate.

The solution to (6.1)–(6.2) will be by Galerkin's method, which is outlined below for conducting walls, $m \neq 0$. The following representations are assumed:

$$w = \mathrm{Re}\left(e^{im\theta}\sum_{n=1}^{N} w_n J_m(\alpha_n^m r)\right)\sin \pi z, \tag{6.3a}$$

$$T = \mathrm{Re}\left(e^{im\theta}\sum_{n=1}^{N} t_n J_m(\alpha_n^m r)\right)\sin \pi z, \tag{6.3b}$$

$$p = \mathrm{Re}\left(e^{im\theta}\sum_{n=1}^{N} p_n R_n^m(r)\right)\cos \pi z, \tag{6.3c}$$

where w_n, t_m, p_n are constants. J_m is the mth-order Bessel function, and, as before,

$$J_m(\alpha_n^m) = 0 \quad (n = 1, 2, \ldots).$$

Thus (6.3a, b) satisfy the necessary boundary conditions. m is the azimuthal wave-number and takes on integer values.

The radial trial functions $R_n^m(r)$ are picked as solutions of the system,

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\left(\frac{m}{r}\right)^2\right)^2 R_n^m = (\lambda_n^m)^4 R_n^m, \tag{6.4a}$$

$$R_n^m = \text{finite} \quad (r=0), \tag{6.4b}$$

$$R_n^m = \frac{dR_n^m}{dr} = 0 \quad (r=1). \tag{6.4c}$$

These functions clearly satisfy the boundary conditions on p. Solutions to (6.4) may be written

$$R_n^m(r) = J_m(\lambda_n^m r) - \frac{J_m(\lambda_n^m)}{I_m(\lambda_n^m)} I_m(\lambda_n^m r), \tag{6.5}$$

if λ_n^m is a root of

$$J_m(\lambda_n^m) I_{m-1}(\lambda_n^m) - J_{m-1}(\lambda_n^m) I_m(\lambda_n^m) = 0. \tag{6.6}$$

Furthermore, the R_n^m are orthogonal,

$$\int_0^1 R_i^m R_j^m r\,dr = \delta_{i,j} J_m^2(\lambda_j), \tag{6.7}$$

where $\delta_{i,j}$ is the Kronecker delta. (For details concerning these functions, see Chandrasekhar 1961, appendix V.) The Bessel functions are also orthogonal, viz.

$$\int_0^1 J_m(\alpha_i^m r) J_m(\alpha_j^m r)\, r\,dr = \frac{J_{m-1}^2(\alpha_j^m)}{2}\,\delta_{i,j}. \tag{6.8}$$

In the usual manner, the finite representations (6.3) are substituted into (6.1) and the residuals are required to be orthogonal to the chosen trial functions, thus generating the linear system of Galerkin equations for the expansion coefficients,

$$p_n\left(\frac{\lambda_n^m}{x_0}\right)^4 J_m^2(\lambda_n^m) = -\pi \sum_{j=1}^N w_j Q(j,n), \tag{6.9a}$$

$$t_n = \left(\frac{x_0}{\alpha_n^m}\right)^2 P w_n, \tag{6.9b}$$

$$\left(t_n - \left(\frac{\alpha_n^m}{x_0}\right)^2 w_n\right)\frac{J_{m-1}^2(\alpha_n^m)}{2} + \pi \sum_{j=1}^N P_j Q(n,j) = 0, \tag{6.9c}$$

for $n = 1, 2, \ldots, N$. Here we have defined the integral

$$\begin{aligned} Q(j,n) &= \int_0^1 J_m(\alpha_j^m r) R_n^m(r)\, r\,dr, \\ &= \frac{-2\alpha_j^m(\lambda_n^m)^2 J_{m-1}(\alpha_j^m) J_m(\lambda_n^m)}{(\alpha_j^m)^4 - (\lambda_n^m)^4}. \end{aligned} \tag{6.10}$$

Combining (6.9) to give one set for the w_n yields

$$\begin{aligned} &\left(P\left(\frac{x_0}{\alpha_n^m}\right)^2 - \left(\frac{\alpha_n^m}{x_0}\right)^2\right) w_n \frac{J_{m-1}^2(\alpha_n^m)}{2} \\ &\qquad = \pi^2 \sum_{i,j=1}^N w_j Q(j,i) Q(n,i) / (J_m^2(\lambda_i^m)(\lambda_i^m/x_0)^4), \end{aligned} \tag{6.11}$$

for $n = 1, 2, \ldots, N$. The system (6.11) is homogeneous and the requirement of non-trivial w_n becomes

$$\det(\mathbf{A}) = 0, \tag{6.12}$$

where $\mathbf{A}$ is an $N \times N$ symmetric matrix with elements

$$A_{i,j} = \pi^2 x_0^4 \sum_{k=1}^{N} \frac{Q(j,k)\,Q(i,k)}{J_m^2(\lambda_k^m)\,(\lambda_n^m)^4} - \left(P\left(\frac{x_0}{\alpha_i^m}\right)^2 - \left(\frac{\alpha_i^m}{x_0}\right)^2\right)\delta_{i,j}. \tag{6.13}$$

Roots of the determinental equation were found numerically for various values of x_0 to yield a modified Rayleigh number–aspect ratio curve. The results of these calculations for $m = 1, 2$, together with those for $m = 0$ (obtained by a slightly different formulation in terms of the Stokes stream function; Homsy 1969) are shown in figure 1.

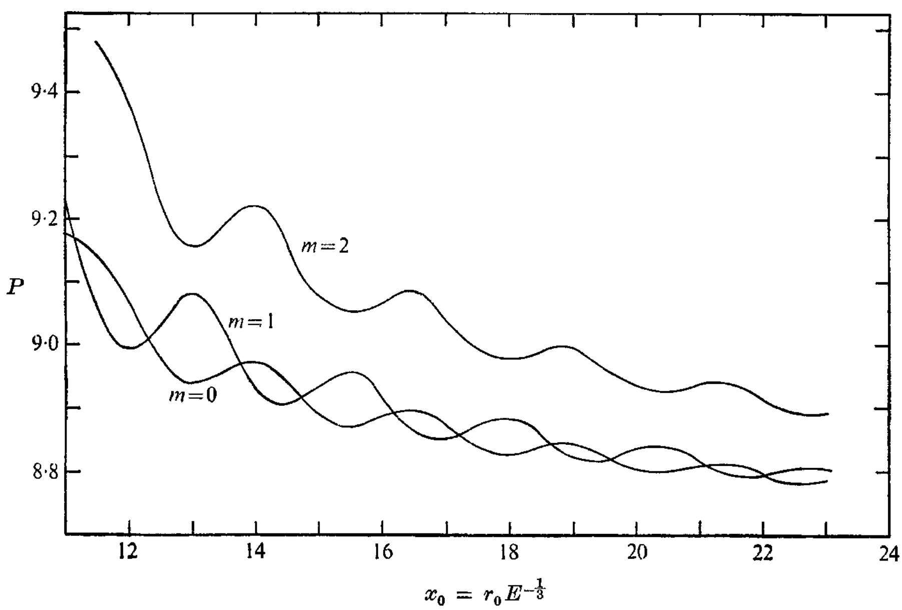

FIGURE 1. Neutral stability curves for three azimuthal wave-numbers.

We note several important features of these results. First, the values of P for various wave-numbers all tend toward the value $P = 8{\cdot}6956$ as x_0 increases, which indicates the diminished constraining effect of the side walls with increasing aspect ratio or decreasing Ekman number. Secondly, each neutral stability curve exhibits local maxima and minima, which are due to cell transitions as the 'comfortable' values of x_0 given by (5.7) are encountered. Similar behaviour has been noted in the case of slow rotation (high Ekman number; Homsy 1969). Lastly, we note that the inclusion of side-wall effects in the linear analysis is in general insufficient to differentiate among the azimuthal wave-numbers. However, for moderate values of x_0, $m = 0$ and 1 are preferred over $m = 2$. Calculations for insulated walls agreed with these results to within 0·1 %, and hence are not given here.

These remarks are of course limited to the asymptotic case, $E < O(10^{-6})$, and are not applicable to moderately small E. Whether linear theory, including the effect of lateral walls, is adequate to predict preferred modes in actual experiments such as those of Koschmider (1967*b*) remains an open question. The above analysis does indicate, however, that axisymmetric disturbances may be assumed in treating asymptotic cases without incurring serious error in the calculated Rayleigh number. We make use of this fact in the subsequent paper which includes the effects of a centrifugally driven basic state.

Grateful acknowledgement is made to the National Science Foundation for partial support of this research through grant number NSF GK 2505. Also, one of the authors (G. M. Homsy) held an NSF Traineeship.

Appendix

To the third order, the interior equations governing the stability of the layer may be written

$$\left.\begin{aligned} \frac{\partial w_i}{\partial z} &= -\nabla_1^4 p_{i+2} \\ \frac{\partial p_{i+2}}{\partial z} &= \nabla_1^2 w_i + T_{i+2} \\ -\left(\sum_{n=0}^{i} P_n w_{i-n}\right) &= \nabla_1^2 T_{i+2} \end{aligned}\right\} \quad (i = 0, 1, 2), \tag{A 1}$$

with the conditions

$$\left.\begin{aligned} w_0 &= 0, \quad z = 0, 1, \\ w_1|_{z=\frac{1}{2}\pm\frac{1}{2}} &= \mp \nabla_1^2 p_2/\surd 2|_{z=\frac{1}{2}\pm\frac{1}{2}}, \\ w_2|_{z=\frac{1}{2}\pm\frac{1}{2}} &= \mp \nabla_1^2 p_3/\surd 2|_{z=\frac{1}{2}\pm\frac{1}{2}}. \end{aligned}\right\} \tag{A 2}$$

Eliminating p_i, T_i, (A 1) give

$$0 = \frac{\partial^2 w_i}{\partial z^2} + \nabla_1^6 w_i - \sum_{n=0}^{i} (P_n \nabla_1^2 w_{i-n}) \quad (i = 0, 1, 2). \tag{A 3}$$

If we now introduce the notation,

$$w_i = W_i(z) J_m(\alpha x) e^{im\theta},$$
$$D = d/dz,$$

(A 3) become
$$0 = D^2 W_i - \alpha^6 W_i - \alpha^2 \sum_{n=0}^{i} P_n W_{i-n}, \tag{A 4}$$

and the boundary conditions become

$$\left.\begin{aligned} W_0 &= 0, \quad z = 0, 1, \\ W_1|_{z=\frac{1}{2}\pm\frac{1}{2}} &= \mp (\surd(2)\alpha^2)^{-1} DW_0|_{z=\frac{1}{2}\pm\frac{1}{2}}, \\ W_2|_{z=\frac{1}{2}\pm\frac{1}{2}} &= \mp (\surd(2)\alpha^2)^{-1} DW_1|_{z=\frac{1}{2}\pm\frac{1}{2}}. \end{aligned}\right\} \tag{A 5}$$

Now let the nth eigenvalue and the nth eigenfunction be denoted by a superscript. Thus,

$$p^n = p_0^n + E^{\frac{1}{6}} p_1^n + E^{\frac{1}{3}} P_2^n + \ldots,$$
$$w^n = w_0^n + E^{\frac{1}{6}} w_1^n + E^{\frac{1}{3}} w_2^n + \ldots + \text{(Ekman correction fields)}.$$

We require that the eigenfunctions be normalized, viz.

$$\langle w^n, w^n\rangle = \int_0^1 w^n w^n dz = 1. \tag{A 6}$$

Here w is taken to mean the total axial velocity, i.e. interior plus Ekman fields.

For the first order, the solution to (A 4) yields

$$W_0^m = \sqrt{2}\sin(m\pi z),$$

$$P_0^m = ((m\pi)^2+\alpha^6)/\alpha^2,$$

and

$$\langle W_0^n, W_0^m\rangle = \delta_{m,n}.$$

Here we have used the normalization requirement (A 6).

The second-order eigenvalue is found from the differential equation,

$$P_0^n\alpha^2 W_1^n + P_1^n\alpha^2 W_0^n = -D^2 W_1^n + \alpha^6 W_1^n. \tag{A 7}$$

We take the inner product of (A 7) with W_0^m and integrate over z, by parts when necessary, to obtain

$$P_1^n\alpha^2\delta_{m,n} = W_1^n D W_0^m\Big|_0^1 + (P_0^m\alpha^2 - P_0^n\alpha^2)\langle W_1^n, W_0^m\rangle. \tag{A 8}$$

For $m = n = 1$, use of (A 5) yields the result in the text, (5.11)–(5.12),

$$P_1^1\alpha^2 = -\frac{1}{\sqrt{(2)}\alpha^2}[(DW_0^1(1))^2 + (DW_0^1(0))^2]$$

$$= -\frac{4\pi^2}{\sqrt{(2)}\alpha^2}. \tag{A 9}$$

Since the $\{W_0^n\}$ are complete, the W_1^n have expansions

$$W_1^n = \sum_{m=1}^{\infty} a_{n,m} W_0^m, \tag{A 10}$$

which converge to W_1^n everywhere except at the end-points. Furthermore,

$$a_{n,m} = \langle W_1^n, W_0^m\rangle.$$

For $n \neq m$, we have from (A 8),

$$a_{n,m} = \frac{2}{\sqrt{(2)}\alpha^2}\frac{nm((-1)^{n+m}+1)}{m^2-n^2}, \quad (m \neq n). \tag{A 11}$$

For $m = n$, use of the normalization requirement (A 6), to $O(E^{\frac{1}{2}})$ implies

$$\langle w_1^n, w_0^n\rangle + \langle \tilde{w}_1^n, w_0^n\rangle = 0. \tag{A 12}$$

It can be shown that the integral involving the Ekman fields $\tilde{w}_1$ is at most $O(E)$, due to the fact that w_0 vanishes near the boundaries, so that (A 12) implies

$$a_{n,n} = 0.$$

Below we shall need a relation for DW_1^1 which cannot be obtained by differentiation of (A 10) (the resulting series is divergent). We rather obtain the result by integrating (A 7) once, using (A 10), (A 11) when necessary. In this manner,

$$DW_1^1 = -\frac{4\pi}{\alpha^2}\left[\cos(\pi z) - \sum_{k=1}^{\infty}\frac{\cos(2k+1)\pi z}{(2k+1)^2-1}\right]. \tag{A 13}$$

For the third-order term, we have from (A 4),

$$\alpha^2(P_0^n W_2^n + P_1^n W_1^n + P_2^n W_0^n) = \alpha^6 W_2^n - D^2 W_2^n. \quad \text{(A 14)}$$

Proceeding as before, the inner product of (A 14) with W_0^m yields, after integration by parts,

$$P_2^n \alpha^2 \delta_{m,n} = W_2^n D W_0^m\big|_0^1 - P_1^n \alpha^2 \langle W_1^n, W_0^m \rangle.$$

For $m = n = 1$, using the fact that $a_{1,1} = 0$, we have simply

$$P_2^1 \alpha^2 = W_2^1 D W_0^1\big|_0^1$$

$$= -\frac{1}{\sqrt{2}\alpha^2}\left(DW_1^1 DW_0^1\big|_1 + DW_1^1 DW_0^1\big|_0\right). \quad \text{(A 15)}$$

Here we have made use of the boundary condition, (A 5). Inserting our expression for DW_1^1 given by (A 13), we have the final result,

$$P_1^2 = \frac{8\pi^2}{\alpha^4}\left(1 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2 - 1}\right) = \frac{6\pi^2}{\alpha^4}. \quad \text{(A 16)}$$

Thus, through $O(E^{\frac{1}{3}})$, the lowest eigenvalue becomes

$$P = \frac{\pi^2 + \alpha^6}{\alpha^2} - \frac{4\pi^2}{\sqrt{(2)}\alpha^4} E^{\frac{1}{6}} + \frac{6\pi^2}{\alpha^6} E^{\frac{1}{3}} + \dots. \quad \text{(A 17)}$$

This corresponds to the result (5.14) in the text. We now minimize P as a function of α in the usual way:

$$\frac{\partial P}{\partial \alpha} = (-2\pi^2\alpha^{-3} + 4\alpha^3) + \frac{16\pi^2}{\sqrt{(2)}\alpha^5} E^{\frac{1}{6}} - \frac{36\pi^2}{\alpha^7} E^{\frac{1}{3}} = 0. \quad \text{(A 18)}$$

Write the wave-number α at which the minimum P occurs as

$$\alpha = \alpha_0(1 + bE^{\frac{1}{6}} + cE^{\frac{1}{3}}).$$

Inserting into (A 18) and equating powers of E yields

$$\alpha_0^6 = \pi^2/2, \quad b = -4/(3\sqrt{(2)}\alpha_0^2), \quad c = -\alpha_0^{-4}.$$

For the value of P at this minimum,

$$P = 3\alpha_0^4\left(1 - \frac{8}{3\sqrt{(2)}\alpha_0^2} E^{\frac{1}{6}} + \frac{4}{9\alpha_0^4} E^{\frac{1}{3}}\right).$$

Inserting the numerical value of α_0 yields the formulae (5.15) in the text.

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